Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3831-3844

http://www.ripublication.com/gjpam.htm

A Logical Approach Proof In Mathematics Education

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Abstract

This paper will presents examples of proofs from the mathematics curriculum and discusses their role in conveying mathematical knowledge.. Educators have long recognized the explanatory value of many proofs, but they have had in mind primarily the light such explanatory proofs can shed on the mathematical subject matter with which they deal. This paper aims to show that proofs can also be bearers of mathematical knowledge in the classroom in another sense. The author will present an logically approach to prove mathematical problems in the learning process. This paper provide two examples of cases; algebra and geometry problems. In both cases, the author describes a method so that students are able to understand the problem with ease. So that students no longer have to memorize formulas so complicated anymore.

Keywords: Proof, learning process, logically approach.

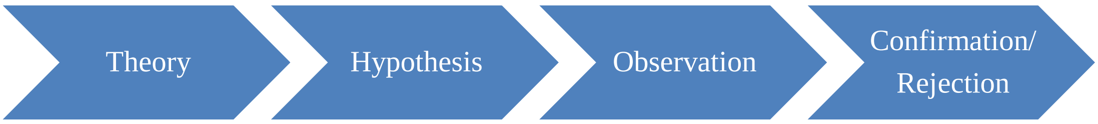
1. INTRODUCTION

The difficulties of pupils experience in arithmetic may be traced to a variety of causes. Numerous studies have been conducted which reveal certain of the typical errors and difficulties pupils encounter in arithmeticMathematics is often considered as a study of arithmeti. These studies have led some teachers of arithmetic to attempt to improve their teaching procedures (Horn, 1941). Other investigations have shown that inadequate study habits, rather than failure to master subject matter, are often the cause of failure to achieve in arithmetic, and that it is necessary to pay as much attention to the development of effective study habits as to the procedure for presentation of instructional materials (Morton, 1953). Further, it now appears that there is not much transfer of learning from one situation to another - while many things are learned simultaneously, every one of these learnings must be oriented toward the needs, interests, and problems of the learner.

Mathematics also referred as a deductive science. This means that the process must be deductive mathematical. Math does not accept generalizations based on observation (inductive), but should be based on deductive proof. Although to help thinking, in the early stages we often need the help of specific examples or illustrations of geometric.
It should also note that both the content and method of searching for the truth in mathematics is different from the natural sciences, especially with science in general. Methods used by the search for truth that mathematics is a deductive science, whereas by natural science is inductive/experiment. But in mathematics, the search for truth can be started by way of inductive, but further generalization is right for a state must be proven deductively. In mathematics, a generalization, natural sciences, theory or proposition that can not be accepted as true before it can be proven deductively.

2. DEDUCTIVE APPROACH

Wilson (2007) said that a deductive approach is concerned with developing a hypothesis (or hypotheses) based on existing theory, and then designing a research strategy to test the hypothesis.
The deductive approach can be explained by the means of hypotheses, which can be derived from the propositions of the theory. In other words, deductive approach is concerned with deducting conclusions from premises or propositions. Deduction begins with an expected pattern that is tested against observations, whereas induction begins with observations and seeks to find a pattern within them, Babbie (2010).
It has been stated that “deductive means reasoning from the particular to the general. If a causal relationship or link seems to be implied by a particular theory or case example, it might be true in many cases. A deductive design might test to see if this relationship or link did obtain on more general circumstances” (Gulati, 2009).
In other words, when a deductive approach is being followed in the research the author formulates a set of hypotheses that need to be tested. Then, through implementation of relevant methodology the study is going to prove formulated hypotheses right or wrong.



In meanwhile, Beiske (2007) informs that deductive research approach explores a known theory or phenomenon and tests if that theory is valid in a given circumstances. “The deductive approach follows the path of logic most closely. The reasoning starts with a theory and leads to a new hypothesis. This hypothesis is put to the test by confronting it with observations that either lead to a confirmation or a rejection of the hypothesis” (Snieder and Larner, 2009).

3. PROOFING IN TEACHING PROCESS

Proof and its teaching have been extensively discussed for the last two decades in the literature on mathematics education and in particular in the proceedings of the International Group for the Psychology of Mathematics (Mariotti 2006). But Rav's (Rav 1999) specific idea, that proof is a bearer of mathematical knowledge, has not been explicitly discussed. The research on proof in mathematics education seems to have dealt primarily with the logical aspects of proof and with the problems encountered in having students follow deductive arguments.

These areas of emphasis are apparent from the specific issues addressed in much of this recent research. The following is by no means an exhaustive list of issues, but is fairly representative: The epistemological aspects of proof (Balacheff 2004; Hanna 1997); the cognitive aspects of proof (Tall 1998); the role of intuition and schemata in proving (Fischbein 1982, 1999); the relationship between proving and reasoning (Yackel and Hanna 2003; Maher and Martino 1996); the usefulness of heuristics for the teaching of proof (Reiss and Renkl 2002); the emphasis on the logical structures of proofs in teaching at the tertiary level (Selden and Selden 1995); proof as explanation and justification (Hanna 1990, 2000; Sowder and Harel 2003); proof and hypotheses (Jahnke 2007); curricular issues (Hoyles 1997); proof in the context of dynamic software (Jones et al. 2000; Moreno and Sriraman 2005); the analysis of mathematical arguments produced by students (Inglis et al. 2007); the relationship between argumentation and proof (Pedemonte 2007). Understandably, the empirical classroom research on the teaching of proof has focused upon students' difficulties with learning proof and on the design of effective teachers' interventions (see the survey of research in the last 30 years in Mariotti 2006).

There are some exceptions to the emphases mentioned above. Lucast (2003) presents a case for "Proof as method: a new case for proof in mathematics curricula," in which it is argued that "proof is valuable in the school curriculum because it is instrumental in the cognitive processes required for successful problem solving" (p. 1). Lucast maintains that proof and problem solving are largely the same process and that both lead to "understanding," and her emphasis is on models of problem solving and their bearing on justification. The present paper, on the other hand, aims to show that in mathematics education a proof can be used to teach mathematical methods and strategies.

Bell (1976) and de Villiers (1990) discussed various meanings and functions of proof. De Villiers (1990, p. 18) listed five functions that he described as "… a slight expansion of Bell's (1976) original distinction between the functions of verification, illumination and systematization." These functions are (bold and italics in the source): "(1) *verification* (concerned with the truth of a statement), (2) *explanation* (providing insight into why it is true), (3) *systematization*, (the organization of various results into a deductive system of axioms, major concepts and theorems), (4) *discovery* (the discovery or invention of new results) and (5) *communication* (the transmission of mathematical knowledge)." This list stopped short of stating that proof contains techniques and strategies useful for problem solving, as Rav claims.

The following two examples deal with proofs that are common to most secondary-school mathematics curricula around the world. They are presented as case studies, in which the proofs have been annotated extensively to demonstrate that they do have the capacity to expand the students' toolbox of techniques and strategies for problem solving.

4. PROBLEM

One of the major criticisms of the traditional curriculum is that students learn to do mathematics by rote, by memorizing procedures and proofs. It is the contention of the advocates of the modern mathematics curriculum that when the subject is taught logically, when the reasoning behind steps is revealed, students will no longer have to rely upon rote learning. They will understand the mathematics. The logical approach is, in other words, also the pedagogical approach and the panacea for the difficulties students have had in learning mathematics.

Just what does the logical approach mean? Basically it is the one commonly used in the traditional curriculum to teach high school geometry. That is, one starts with definitions and axioms and proves conclusions, called theorems, deductively. Though this approach has been used in geometry, it has not been used in the teaching of arithmetic, algebra, and trigonometry. Hence, so far as this feature of the new curriculum is concerned, the major change is in these latter subjects. Let us see what the deductive approach to arithmetic and algebra entails.

4.1. Case I: The Quadratic Formula

While students at school get exposed to very few "theorems," particularly in areas other than geometry, they nevertheless have to learn a few formulae, which are essentially statements of results. An example of this is the formula for the solution of a quadratic equation.

The solutions of the quadratic equation

$$ax^2 + bx + c = 0 \text{ where } a \neq 0, \text{ are given by } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

At the most basic level, the students may simply use this formula to solve particular quadratic equations. It is even possible for them to apply it blindly, not realizing that they can check their solutions by substituting back into the equation. However, if they do make such substitutions, then, on empirical grounds, they will undoubtedly come to trust it and apply it mechanically.

At this point, students may perceive that there are two independent methods of solving quadratic equations, one, factoring, that is not guaranteed of success, and the other, use of the formula, which will work all of the time.

One way to establish the formula is to substitute the values of x given by the formula and verify that they do indeed satisfy the quadratic equation. This is a legitimate proof, but does it leave anything to be desired? On the plus side of the ledger, it emphasizes what the formula actually delivers: values of the variable that satisfy the equation. On the minus side, apart from the messiness of the substitution, how likely is it that students will be able to apply it flexibly and reliably? There is no indication of the significance of the formula, how such a complicated expression might arise, and how it might fit in with other properties and applications of the quadratic and related functions. The formula is a black box. Simply verifying that the formula works has another defect: We do not know that it yields the only solutions of the quadratic equation. There may be other numbers that satisfy it, and perhaps we may come across a situation in which these alternatives are what we want.

An actual discussion of how the formula is obtained leads us to questions of strategy. In the present case, we might frame the question differently. Instead of asking, “What is a formula for the solutions of a quadratic equation?” we ask, “How can we solve a quadratic equation?” The second question induces us to think about process rather than product, and to consider how we might start.

For example, we might ask whether there are quadratic equations that are easy to solve. There are two possible answers that we might give. First, we can solve equations when the quadratic is factorable into linear polynomials. Secondly, we can solve quadratic equations of the form $x^2 = k$, when k is positive; indeed, in this case the answer is: $x = \pm k$. Is there any way we can reduce the problem of solving a general quadratic to either of these cases? We note that in fact these are related; the equation $x^2 = k$ can be converted to $0 = x^2 - k = (x - k)(x + k)$. (Note: It may be necessary in some circumstances to satisfy students that $x^2 = k$ has only these two solutions. This might be done by considering the monotonicity of the function x^2 or by appealing to the fact that the product of two nonzero quantities cannot vanish. Either way inducts students into the underlying structure.)

Most students will probably not know how to proceed from here on their own, and will have to be taught the technique of completing the square. But such considerations will inform the technique when it is presented. What makes it easy to solve $x^2 - k = 0$ is the absence of the linear term, and so we need to perform a gambit in effect to absorb the linear term in the general equation. The key recognition is that $ax^2 + bx$ can be rewritten as $a(x^2 + \frac{b}{a}x\ )$ and that the quantity in parenthesis comprises the first two terms in the expansion of $\left(x + \frac{b}{2a}\right)^2$ and differs from this expansion by a constant, namely, $\frac{b^2}{4a^2}$. Thus we *“complete the square”*; add a term on the left side to give us the square of a linear polynomial, and then subtract it again, in effect adding 0. When $\neq 0$, we transform $ax^2 + bx + c = 0$ to:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

and finally arrive at the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a \neq 0$.

This may be the first time that secondary school students see this general technique of adding and then subtracting a term in an expression, a useful technique that they will see frequently as they advance their study of mathematics. We note here that completing the square does not stem logically from a previous statement or axiom. Rather it is a topic specific move and an additional mathematical tool for the students to use in other similar situations.

By adding this technique to their toolkit, students may be able to take advantage of situations where it is more efficient to use this technique rather than to simply apply the formula. For example, given the task of solving $x^2 - 8x - 48 = 0$, and not recognizing a factorization, the student could just as easily render the equation as $(x-4)^2 - 64 = 0$ as apply the formula.

Having explicitly identified the ingredients of the situation, we can play around with them. Both factoring quadratics and using the formula lead to solutions of the equation. But we can use the formula also to *obtain* a factorization for any quadratic, even if the coefficients have to be non-integers. Since students going on in mathematics will inevitably meet situations, other than solving equations, in which factoring a polynomial is desirable, we have to be sensitive to possible procedures for this. Even more useful than the formula itself is the strategy – completing the square.

The following example will illustrate.
Consider the quartic polynomial: $x^4 + 4$

Is this factorable over the integers? It is not obvious that it is. However, if students have been able to absorb the essence of the square-completion technique, then some might be able to complete the square in a different way to get

$$(x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$$

There is the possibility of students being able to leap ahead in the curriculum. The equation $x^4 + 4 = 0$ would normally require some knowledge of complex numbers and roots of unity to solve; however, from the above factorization as a product of quadratics, even a student in the lower secondary grades would be able to generate a solution.

We might also ask, if we can complete the square, why not complete the cube, and apply an analogous technique to solving

$$ax^3 + bx^2 + cx + d = 0$$

The left side can be written as

$$a\left(x+\frac{b}{3a}\right)^3+\left(c-\frac{b^2}{3a}\right)x+\left(d-\frac{b^3}{27a^3}\right)=0$$

In this way, we can reduce the problem of solving any cubic to solving cubics of the form $x^3 - px + q = 0$, which is the usual starting point for general methods of solving the cubic. In a similar way, we can arrive at $x^4 + ax^2 + bx + c = 0$ as a "canonical form" for equations of the fourth degree.

If we follow the invitation of the proof to consider equations of the third and fourth degrees, we realize that we have developed means of expressing the roots in terms of the coefficients, using the four arithmetic operations along with the extraction of square and cube roots. It is a natural question to ask whether the solutions of higher degree equations are attainable from the arithmetic operations and extraction of roots of any order applied to the coefficients.

Delving into the proof reinforces an important perception that students should have about algebra. In any algebraic quest, we are in the business of reading off information from an expression. Sometimes the information can be easily read off, and sometimes it is buried and needs to be brought to light. The purpose of algebraic manipulation is to cast an expression into a form in which the desired information can be drawn. In the case of a quadratic, we have the standard form in descending powers of the variable, the factored form as a product of linear factors and the completion of the square. The factored form allows us to immediately read off its roots. When we use the completion of the square form, as shown above, while we need an additional step to solve the equation, we can see right away where the quadratic polynomial assumes its maximum or minimum value and exactly what that value is. In fact, we do also get some information about the roots as well. If both a and $4ac - b^2$ are positive, for example, then we can see that the quadratic is positive for all real values of x and so has no real roots.

Thus we see that consideration of the proof has benefits that go far beyond the mere validation of a formula. In the present case, we gain the perception of reducing the general situation to a canonical type, the understanding of how the character of the roots depends on the coefficients, the certainty that the quadratic equation can have no more that two roots. More importantly from the point of view of this paper, we gain the knowledge of a technique whose range of applicability is wider than the situation at hand, and a broader knowledge of quadratics that can be knitted into a more comprehensive whole.

4.2. Case II: Does An Angle Inscribed in a Semi-circle is a Right Angle ?

The various proofs of this theorem will highlight the mathematical knowledge they contain. In addition, they show mathematical results as markers on a path, ways of giving form to a mathematical journey. A proof tells us where a mathematical result lives, about its neighborhood and associates; it highlights the significant ideas that underlie it.

Proposition. Let *A* and *B* be opposite ends of the diameter of a circle and let *C* be a point on its circumference. Then angle *ACB* is right.

This geometric result is familiar to many high school students. Although it is simply stated, there are many dimensions to it and the mere statement of the result will inevitably fail to convey its richness. As with any geometric result, certain properties are highlighted for consideration and related; the posited relationship might seem quite mysterious and incomprehensible. In order to feel more at home and perceive that the result is somehow natural, it is desirable to probe deeply and sense how the mathematical structure is woven together. This particular result can be approached from many directions (Barbeau 1988), and the purpose of what follows is to comment on the mathematical content of some of these.

The standard argument makes the observation that with *O* the centre of the circle, *OA*, *OB* and *OC* are all equal and so we have some equal angles in isosceles triangles and draw the conclusion that the angle at *C* is the sum of the angles at *A* and *B*, and so is equal to 90°. This argument highlights the significance of the circle hypothesis – the centre bisects the diameter and is equidistant from *A*, *B* and *C* (see Fig.1).
What are the other ingredients? We need a theorem about isosceles triangles and about the sum of the angles in a triangle. The last raises the question of the sort of geometry in which the result holds. This is a Euclidean result. The standard argument also raises the question of the truth of the converse. Suppose that we have a triangle *ABC* whose right angle is at *C*. Then the angle at *C* is the sum of the angles at *A* and *B*; so we can construct a cevian *CO* which splits the angle at *C* so that angle *ACO* = angle *CAO* and angle *BCO* = angle *CBO*. This gives us a couple of isosceles triangles and so *AO* = *BO* = *CO*. Thus, *C* lies on a circle with centre *O* and diameter *AB*. This proof gives us a diagram that can be interpreted in two ways – one that gives us the result itself and the second that gives us its converse.

Suppose we tweak the diagram of this argument in another way. Produce *CO* to some point *X*, and note that the exterior angle *XOB* is twice angle *OCB* and exterior angle *XOA* is equal to twice angle *ACO* (see Fig. 2). Then the straight angle *AOB* is twice angle *ACB*, making the latter angle right. Looking at the matter in this way reveals that it is part of a larger picture. By bending *AB* at *O*, we can now deduce, with the same argument, the result that angle *ACB* is half the angle subtended at the centre by a chord *AB*, so that the angle subtended by a chord at the major arc of a circle is constant (see Fig. 3).

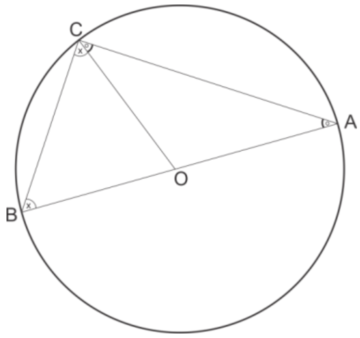


Figure 1: Angle inscribed in a semi-circle

In a similar way, it can be shown that the angle subtended at the minor chord is constant (and supplemental to the other angle). From here, it is a natural step to obtain some properties of concyclic quadrilaterals. This more general result is not contained in the statement of the theorem, but by looking at the elements of the proof, we can arrive at it. The next proof is the second transformation argument that involves a dilatation with factor 1/2 and centre *B*. This dilatation takes $A \to O$ and $C \to E$, the midpoint of chord *CB*. Now, *E* being the midpoint of chord *CB* means that *OE* right bisects it (this is basically a consequence of triangle *COB* being isosceles).

Thus *OE* is perpendicular to *CB*. Now reverse the dilatation; since angles are preserved *AC* is perpendicular to *CB*, and we are done. This argument has quite a different flavor than the first one and introduces a symmetry element into the situation that is not apparent from the bald statement of the theorem. Thus the proof contains mathematical knowledge beyond mere deductive reasoning. There are some areas of mathematics, such as algebra, calculus and trigonometry that provide a general framework for proving results of a particular type. In using

general techniques, we are situating the result among a category of those that can be handled in a specific way. This focuses attention on the particular characteristics that make the techniques applicable. For example, we can conceive of the situation of the proposition in the cartesian plane, the complex plane or two-dimensional vector space (see Fig. 4). The proposition contains elements that are capable of straightforward formulation in each of these areas.

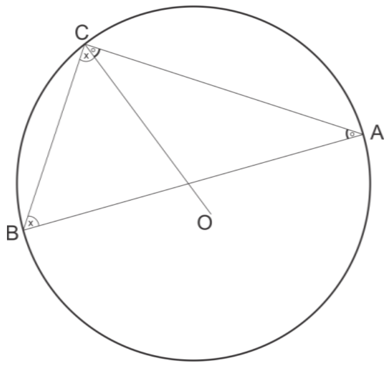


Figure 2: Extended CO

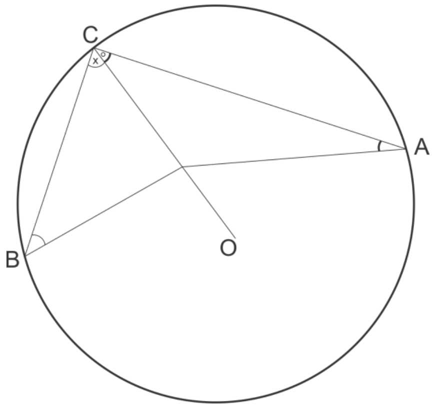


Figure 3: Bending AB at O

In the cartesian plane, the circle can be described by a simple quadratic equation and the condition for perpendicularity of two lines involves their slopes. If we coordinatize A, B and C as $(-1,0),(1,0)$ and (x,y) where $x^2+y^2=1$, then we can check that 1 plus the product of the slopes of AC and BC is 0. In the complex plane, where multiplication by i corresponds to the geometric rotation through 90° about the origin, the proof becomes a matter of verifying that if A is taken to be −1, B as +1 and C as z where $\overline{zz}=1$ then $(z-1)/(z+1)$ is a real multiple of $z-\bar{z}$ and so pure imaginary. Finally, the vector proof can be carried out with or without coordinates. In the latter case, the proof is particularly slick. Taking the centre of the circle as the origin of vectors, then $(C-B)\cdot(C-A)=C^2\ -C.(A+B)+A.B=0$ since $A=-B$ and $C^2=B^2=A^2$ is the square of the radius of the circle.

Some proofs reveal more than others; from some of the arguments, it can be quickly inferred that angle *ACB* is right if and only if *AB* is the diameter of a circle that contains *C*, so that the converse really is also built into the proof.

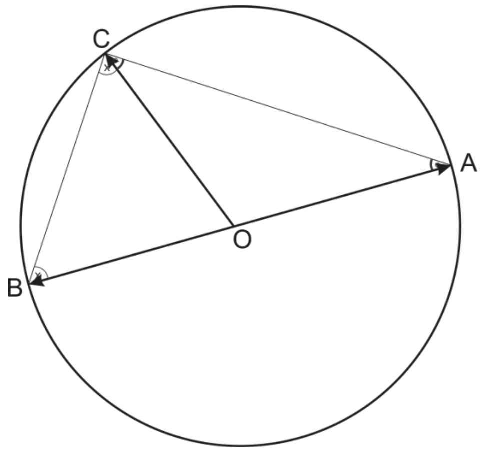


Figure 4: Vector argument

In summing up the lesson of these case studies, one might consider that those students whose learning is most robust are likely to be those who have developed a multifaceted way of looking at mathematical facts. Their mathematical knowledge is rich with many connections and corroborations. One way of presenting our point in this paper is to say that the bald statement of results and practice of techniques in the classroom has little chance to foster this multifaceted view, while having to construct or follow well-chosen proofs, with the concomitant exposure to unfamiliar methods, tools, strategies and concepts that Rav has shown, can convey to the student a much richer understanding of mathematics.

Several additional examples could have been presented, such as the many different proofs of the infinitude of primes, each resting on a particular technique; the hundreds of proofs of the Pythagorean theorem, each using a different method or technique; the many proofs of numerical results that may be proved by mathematical induction or by an algebraic technique such as the telescoping method.
An example of the last is the finite sum of the series

$$\sum_{n=1}^{N} \frac{1}{n(n+1)}$$

which can be treated as a telescoping sum, as follows:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{N} + \frac{1}{N}\right) - \frac{1}{N+1} = 1 - \frac{1}{N+1}$$

CONCLUSION

As discussed in this paper shown that proofs can extend mathematical knowledge by bringing to the fore new techniques and methods, and it has maintained in fact that for this reason proofs should be a primary focus of interest in mathematics. Argue that what is true of mathematics itself may well be true of mathematics education: In other words, that proofs could be accorded a major role in the secondary-school classroom precisely because of their potential to convey to students important elements of mathematical elements such as strategies and methods.
It is important to call attention to the potential for exploiting this aspect of proof in the classroom. Mathematics educators have always made use of the fact that there are many different styles of proving, showing students how one can arrive at valid conclusions in different ways, using topic-specific moves, algebraic manipulations,

geometric concepts, dynamic geometry, arithmetical computations, computing and more. Nevertheless, educators have overlooked to a large extent the role of proof as a bearer of mathematical knowledge in the form of methods, tools, strategies and concepts that are new to the student and add to the approaches the student can bring to bear in other mathematical contexts.

The adoption of the approach to proof which we have presented would require that proofs suitable for this teaching approach and for the secondary-school curriculum be assembled and polished and then be made available to teachers and curriculum planners. It would also necessitate research into the most effective ways to teach proofs with this new approach in mind. The approach to using proof which we have discussed here does not challenge in any way the accepted “Euclidean” definition of a mathematical proof (as a finite sequence of formulae in a given system, where each formula of the sequence is either an axiom of the system or is derived from preceding formulae by rules of inference of the system), nor does it challenge the teaching of Euclidean derivation itself. It points out, however, that the teaching of proof also has the potential to convey to students other important pieces of mathematical knowledge and to give them a broader picture of the nature of mathematics. In highlighting a sometimes unappreciated value of proof, it also gives educators an additional reason for keeping proof in the mathematics curriculum.

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